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NONLINEAR HEAT-CONDUCTION PROBLEM FOR AN ORTHOTROPIC THERMOSENSITIVE PLATE

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We consider the solution of the nonlinear heat-conduction problem for an orthotropic thermosensitive plate heated by a uniformly moving heat source.

Glass-reinforced plastics are widely used in machines and apparatuses. Owing to good heat resistance structural elements of glass-reinforced plastics heat up rather slowly, which is very important under the conditions of high-gradient heat fluxes. The inner layers of parts made of glass-reinforced plastics are not warmed up and remain strong enough over the time needed for the operation of the structure even at surface temperatures of several thousands degrees Celsius.

For the most part, heat-conduction problems for heatproof structures made of anisotropic materials are considered by researchers under the assumption that the thermophysical characteristics of these materials are temperature-independent. This is explained by both the complexity of obtaining an exact analytical solution for a nonlinear heat conduction problem and the difficulties in determining experimentally the dependences of the thermal parameters of the material (thermal conductivity, volumetric heat capacity, etc.) on temperature and, consequently, their uncertainty.

In the present work we give a solution of the heat conduction problem for an orthotropic plate starting from the premise that the thermophysical characteristics of the material depend on temperature (thermosensitive material). We also analyzed the effect exerted by the orthotropy degree and other factors on the behavior of the temperature field.

Suppose a semi-infinite orthotropic thermosensitive plate is heated by a concentrated heat source with intensity w_0 that moves uniformly and parallel to the edge $x_0 = 0$ of the plate in the positive direction of the $0y_0$ axis with a certain constant velocity v. Heat transfer between the plate surface $x_0 = 0$ and the medium of constant temperature t_m follows Newton's law; heat losses from the side surfaces are neglected [1].

We assume that the temperature distribution is quasistationary with respect to the moving coordinate system. In this case, the boundary-value problem for the temperature field resulting from a given thermal effect in the moving coordinate system x0y ($x = x_0$, $y = y_0 - v\tau$) takes the form

$$\frac{\partial}{\partial x}\left[\lambda_x(T)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[\lambda_y(T)\frac{\partial T}{\partial y}\right] = -c_v(T)\,v\frac{\partial T}{\partial y} - w_0\delta(x-d, y), \tag{1}$$

$$T = T_0, \quad \frac{\partial T}{\partial y} = 0 \quad \text{for} \quad |y| \to \infty, \tag{2}$$

$$\lambda_x(T)\frac{\partial T}{\partial x} = \alpha(T - t_m) \quad \text{for} \quad x = 0, \tag{3}$$

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where $\delta(\xi, \eta) = \delta(\xi)\delta(\eta)$; $w_0 = q/2\delta$; d is the distance of the heat source from the plate edge; T_0 is the reference temperature.

Assuming that the temperature dependence of the thermal conductivities λ_x and λ_y and heat capacity c_v is the same in character, i.e., $\lambda_y(T)/\lambda_x(T) = k_y \cong$ const; $\lambda_x(T)/c_v(T) = a \cong$ const, it is possible to partially linearize the nonlinear boundary-value problem (1)-(3) by introducing the Kirchhoff variable

$$\theta = \frac{1}{\lambda_0} \int_{T_0}^{T} \lambda_x(\xi)\, d\xi \tag{4}$$

After transformations, we obtain

$$\frac{\partial^2 \theta}{\partial x^2} + k_y \frac{\partial^2 \theta}{\partial y^2} = -2\omega \frac{\partial \theta}{\partial y} - Q_0 \delta(x - d, y), \tag{5}$$

$$\theta = 0, \quad \frac{\partial \theta}{\partial y} = 0 \quad \text{for} \quad |y| \to \infty, \tag{6}$$

$$\lambda_0 \frac{\partial \theta}{\partial x} = \alpha (T - t_m) \quad \text{for} \quad x = 0, \tag{7}$$

where $\omega = v/2a$; $Q_0 = w_0/\lambda_0$; λ_0 is the reference value of the thermal conductivity coefficient λ_x.

Boundary condition (3) involves the value $T(0, y)$, i.e., the solution of the nonlinear problem sought, taken on the boundary surface. Since $T(0, y)$ is a function of the coordinate y alone, we approximate it by asymmetric unit functions [2] in the following way:

$$T(0, y) = T_0 + \sum_{i=1}^{m} T_i \left[S_-(y - y_i) - S_-(y - y_{i+1})\right], \tag{8}$$

where T_i $(i = \overline{1, m})$ are unknown values; $y_i \in I_y = \{y\colon y \in R\}$.

With allowance for Eq. (8), boundary condition (7) can be written as follows:

$$\lambda_0 \frac{\partial \theta}{\partial x} = \alpha \sum_{i=1}^{m} T_i \left[S_-(y - y_i) - S_-(y - y_{i+1})\right]. \tag{9}$$

In relation (9), without loss of generality, we assumed that $t_m = T_0$.

The solution obtained for boundary-value problem (5), (6), and (9) on the basis of a integral Fourier transform with respect to the coordinate y has the form

$$\theta(X, Y) = \mathrm{Bi} \sqrt{\left(\frac{2}{\pi k_y}\right)} \sum_{i=1}^{m} T_i^* \int_{Y-Y_i}^{Y-Y_{i+1}} \exp(-P\zeta)\, K_0\left(P\sqrt{k_y X^2 + \zeta^2}\right) d\zeta +$$

$$+ \frac{\mathrm{Po}}{2\pi\sqrt{k_y}} \exp(-PY) \left\{K_0\left(P\sqrt{k_y (X + D)^2 + Y^2}\right) +\right.$$

$$\left. + K_0\left(P\sqrt{k_y (X - D)^2 + Y^2}\right)\right\}, \tag{10}$$

where $P = \mathrm{Pe}/k_y$; $X = x/\delta$; $Y = Y/\delta$; $Y_1 = y_i/\delta$; $D = d/\delta$; $T_i^* = T_i/T_0$; $\mathrm{Po} = Q_0/T_0$ is the Pomerantsev number.

Using relation (4), on the basis of Eq. (10) we find the desired temperature field. We note that it is expressed in terms of the unknown coefficients T_i. For their determination we should specify the law of the de-

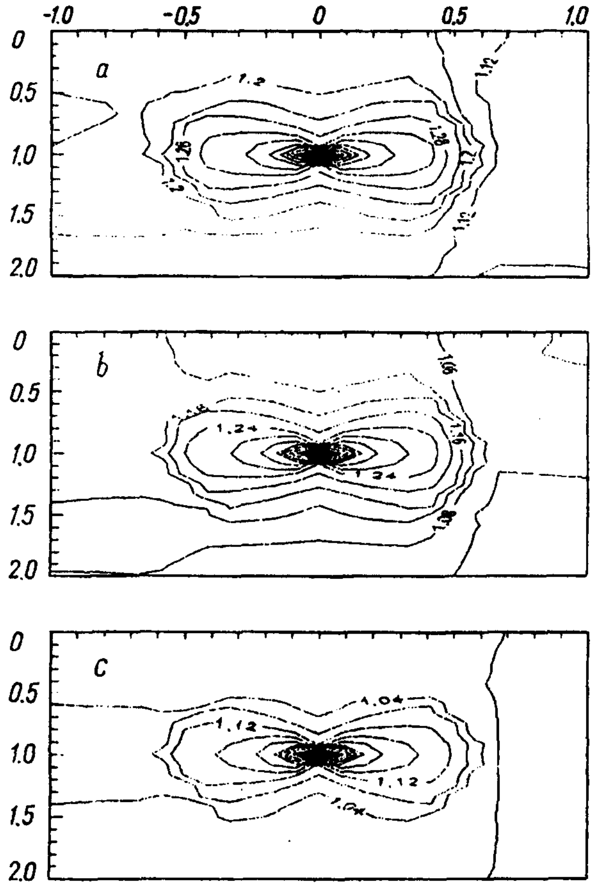

Fig. 1. Lines of level dimensionless temperature T_* of orthotropic thermosensitive system: a) Pe = 0.5; b) 1; c) 5.

pendence of the thermal conductivity coefficients (in this case the thermal conductivity coefficient only in the direction of the 0x axis). The most widely used is the linear dependence, i.e.,

$$\lambda_x(T) = \lambda_0 (kT - 1)\,. \tag{11}$$

Relation (11) is obtained by approximating the thermal conductivity coefficient of glass-reinforced plastic vs. the temperature (from 480 to 1200°C) curve given in [3]; in this case, the reference temperature is equal to 480°C. Then the desired temperature is defined as a positive root of the quadratic equation

$$T_*^2(X, Y) - \frac{2}{kT_0} T_*(X, Y) - \frac{2}{kT_0}\left[\theta(X, Y) + 1 - \frac{\gamma}{2}\right] = 0\,. \tag{12}$$

After some transformations invoking Eqs. (8) and (12), we obtain the following recurrence relations for T_i^*:

$$T_i^* = -\frac{1}{\gamma} + \sqrt{\left(\frac{1}{\gamma^2} + \frac{2}{\gamma\sqrt{k_y}}\,\Omega(Y_i)\right)}\,, \tag{13}$$

where

$$\Omega(Y) = \frac{\text{Po}}{\pi}\Psi(Y) + \sqrt{\left(\frac{2}{\pi}\right)}\ \text{Bi}\sum_{j=1}^{i-1} T_j^*\,[\Phi(Y - Y_{j+1}) - \Phi(Y - Y_j)] + \frac{\gamma}{2} - 1\,;$$

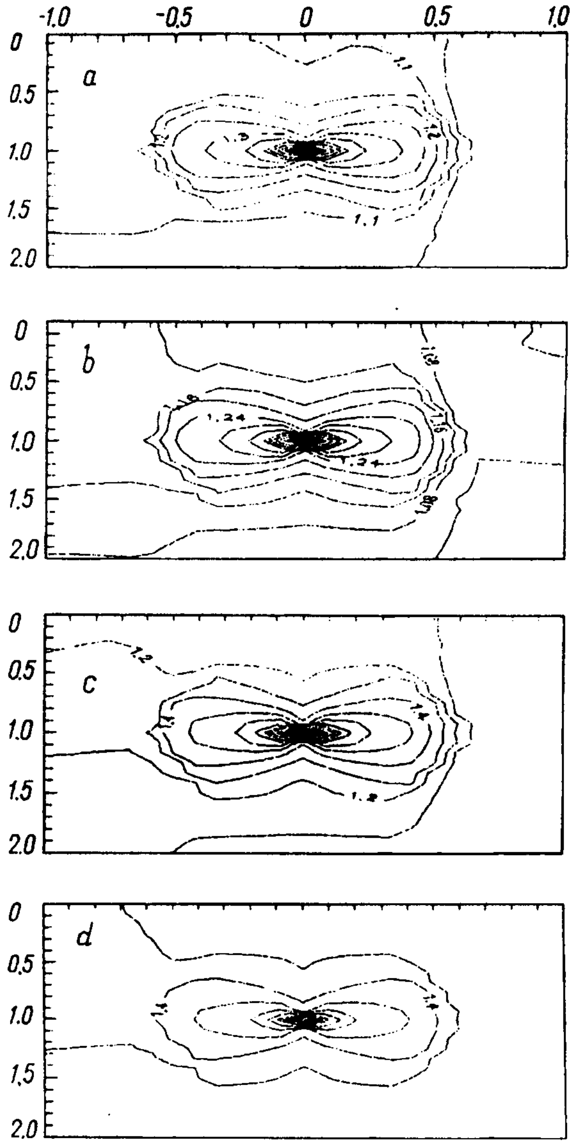

Fig. 2. Lines of level dimensionless temperature T_* constructed at $D = 1$, Pe = 1: a) and b) isotropic and orthotropic thermosensitive systems; c and d) orthotropic and isotropic nonthermosensitive systems.

$$\Psi(Y) = \exp(-PY)\, K_0\left(P\sqrt{k_y D^2 + Y^2}\right);$$

$$\Phi(Y - Y_j) = (Y - Y_j)\exp(-Y_j^*)\,[K_0(Y_j^*) - K_1(Y_j^*)]\,S_-(Y - Y_j);$$

$$Y_j^* = (Y - Y_j)\,P;\ \ \gamma = kT_0;\ \ T_* = T/T_0.$$

Using Eqs. (10) and (13) we obtain a number of particular cases. So, by setting $k_y = 1$ in these relations, we find the solution of the corresponding heat conduction problem for an isotropic thermosensitive system. Assuming in boundary-value problem (1)-(3) that $\lambda_x(T)$ = const and $\lambda_y(T)$ = const and making the necessary transformations, we obtain an expression that determines the temperature field in an orthotropic nonthermosensitive plate:

$$T_*(X, Y) = \frac{\text{Po}}{2\pi\sqrt{k_y}}\exp(-PY)\times$$

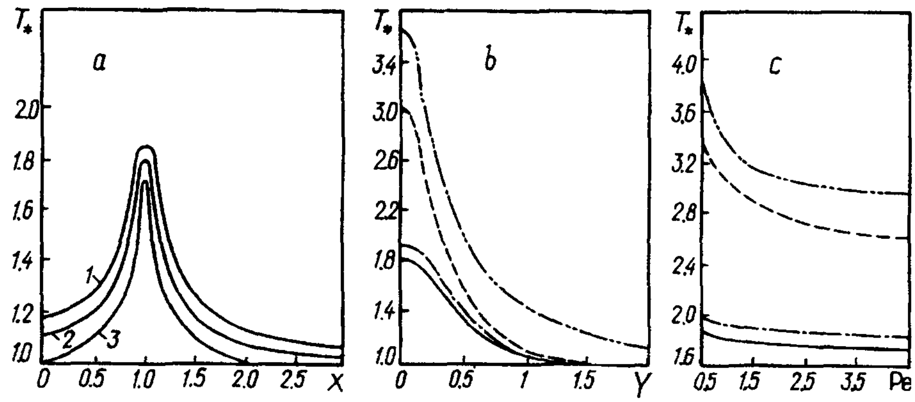

Fig. 3. Dependence of dimensionless temperature T_*: a) on the coordinate X at $Y = 0$ (1) Pe = 0.5; 2) 1; 3) 5; b) on the coordinate Y at $X = D$; Pe = 0.5; c) on the parameter Pe at $D = 1$; $X = 1$; $Y = 0$.

$$\times \left\{ K_0 \left(P \sqrt{k_y (X - D)^2 + Y^2} \right) + \int_0^\infty F(X, \zeta) \cos \frac{\zeta Y}{\sqrt{k_y}} d\zeta \right\} + 1 , \tag{14}$$

where

$$F(X, \zeta) = \exp [- (X + D) \zeta_*] \left(1 - \frac{\text{Bi}}{\zeta_*} \right) (\text{Bi} + \zeta_*)^{-1} ; \; \zeta_* = \sqrt{\text{Pe} P + \zeta^2} .$$

Setting $k_y = 1$ in Eq. (14), we obtain an expression for the temperature field in an isotropic nonthermosensitive system.

For implementation of a solution a set of programs was developed for the IBM PC XT/AT. As initial data for calculations we took: $k_y = 1.6$; $k = 0.494 \cdot 10^{-2}$ $^\circ C^{-1}$; Po = 1; Bi = 0.01; $D = 1$. The plate was assumed to be made of a glass-reinforced plastic. When determining the values of T_i^* using Eqs. (13), convergence was considered to be attained upon compliance with the criterion $|T_{i+1}^* - T_i^*| \le \varepsilon$, where ε is an arbitrarily small number; the computer time for a typical variant, requiring not more than ten iteration steps, is of the order of 10 sec.

The results of numerical investigations are given in Figs. 1-3. The distributions of constant temperature lines (isotherms) for different values of the Pe number are shown in Figs. 1 and 2. The graphs in Figs. 3a and 3b illustrate the temperature change over the coordinates X and Y, while those given in Fig. 3c illustrate the temperature dependence on the Peclet number. The solid curves correspond to the temperature in the orthotropic thermosensitive system; the dashed curves, the isotropic thermosensitive system; the dashed-dotted curves, the orthotropic nonthermosensitive system and the dashed curves with two points, the isotropic nonthermosensitive body.

From the numerical data presented we can see that in the case of a low velocity of the heat source, heat diffusion is considerable in all directions. As v, or, which is the same, Pe, increases, we observe localization of the temperature field in the vicinity ($|X| < D/2$, $|Y| < D/2$) of the heat source.

Allowance for orthotropy leads to an insignificant decrease in temperature compared to an isotropic medium, i.e., of the order of 6–12%. The relative effect of the orthotropy degree is virtually constant for both thermosensitive and nonthermosensitive materials. To a greater extent, an influence on the temperature distribution is exerted by the thermosensitivity of the material; numerical calculations indicate that its consideration leads to a decrease in temperature (by almost a factor of 2) for both isotropic and orthotropic bodies.

Numerical calculations by formula (14) showed that the upper limit of integration can be limited to $\zeta = 10$ without a substantial loss of accuracy. The improper integrals were calculated by the method of cubic splines using the standard QUADPACK program [4]. On the basis of the numerical investigations performed we can assert that the integral with the infinite limit in Eq. (14) at any Bi has an order of magnitude not greater than 10^{-5}, i.e., in investigating the thermal state of a nonthermosensitive orthotropic or isotropic plate, we can use an approximate formula, neglecting the improper integral entering into Eq. (14).

The results obtained can form a basis for selecting regime parameters in investigations of processes of heat conduction in heatproof structural elements and justification for taking into account the variability of the thermal conductivity coefficients in solutions of heat and mass transfer problems.

NOTATION

$\lambda_x(T)$, $\lambda_y(T)$, coefficients of thermal conductivity along the principal axes of orthotropy coinciding with the axes of the 0x and 0y coordinates; α, heat transfer coefficient; q, heat source power; $c_v(T)$, volumetric heat capacity; $S_-(\xi)$, asymmetric unit function; $\delta(\xi)$, Dirac delta-function; 2δ, thickness of plate; $T(x, y)$, temperature field; τ, time; $\mathrm{Bi} = \alpha\delta/\lambda_0$, Biot number; $\mathrm{Pe} = v\delta/2a$, Peclet number; $K_\nu(\xi)$, modified Bessel function of order ν ($\nu = 0; 1$).

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